Journal of Approximation Theory **103**, 357–359 (2000)
doi:10.1006/jath.1999.3430, available online at http://www.idealibrary.com on IDEAL®

NOTE

Where Are the Nodes of "Good" Interpolation Polynomials on the Real Line?

J. Szabados

Alfréd Rényi Mathematical Institute, Hungarian Academy of Sciences, Reáltanoda u. 13-15, H-1053 Budapest, Hungary

Communicated by Paul Nevai

Received July 1, 1999; accepted September 16, 1999

It is shown that the interval where the nodes of a "good" interpolation polynomial are situated is strongly connected with the Mhaskar–Rahmanov–Saff number. © 2000 Academic Press

Let w be a Freud-type weight on **R**. For a formal definition of Freud-type weights see, e.g., [LL]. Here we mention only the archetypal example $w(x)=e^{-|x|^\alpha}$, $\alpha>1$. Let Π_n denote the set of algebraic polynomials of degree at most n, and let $\|\cdot\|$ denote the supremum norm over **R**. It is known that with each Freud-type weight and natural integer n, one can associate a positive number a_n such that

$$\|q_n\| = \max_{|x|\leqslant a_n} |q_n(x)| \tag{1}$$

for all weighted polynomials q_n of degree at most n, that is, for all q_n such that $q_n/w\in\Pi_n$ (see [MS]). The quantity a_n is often called the Mhaskar–Rahmanov–Saff number which tells us where the norm of a weighted polynomial "lives."

Now consider the Lagrange interpolation on arbitrary nodes $x_0<x_1<\cdots<x_n$. The weighted Lebesgue constant plays an important role in the theory of weighted convergence of Lagrange interpolation in certain classes of functions. It is defined as the supremum norm on **R** of the weighted Lebesgue function

$$\lambda_n(x)=w(x)\sum_{k=0}^{n}\frac{|\ell_k(x)|}{w(x_k)},$$

0021-9045/00 $35.00

where

$$\ell_k(x) = \prod_{\substack{i=0 \\ i \neq k}}^{n} \frac{x - x_i}{x_k - x_i}, \qquad k = 0, ..., n,$$

are the fundamental polynomials of Lagrange interpolation. In this note we establish a relation between the size of the nodes and the weighted Lebesgue constant.

PROPOSITION 1. *For every system of nodes we have*

$$\max_{0 \leqslant k \leqslant n} |x_k| \leqslant a_n \left(1 + c \left(\frac{\log \|\lambda_n\|}{n}\right)^{2/3}\right)$$

with some constant $c > 0$ *depending only on* w.

Proof. If $q_n/w \in \Pi_n$ is arbitrary, then

$$|q_n(x)| \leqslant e^{nU_{n,\, a_n}(x/a_n)} \|q_n\|, \qquad x \in \mathbf{R}$$

where $U_{n,\, R}(x)$ is a so-called "majorizing function" (cf. [LL, Lemma 7.1] applied with $R = a_n$ and combined with (1)). Using [LL, inequality (7.14)] with $R = a_n$ and $\varepsilon = (x/a_n) - 1$ we obtain

$$|q_n(x)| \leqslant e^{-c_1 n((x/a_n) - 1)^{3/2}} \|q_n\|, \qquad |x| \geqslant a_n, \tag{2}$$

where $c_1 > 0$ depends only on w.

Now let $y_n \in \mathbf{R}$ be such that $\|\lambda_n\| = \lambda_n(y_n)$, and consider the weighted polynomial

$$q_n(x) := w(x) \sum_{k=0}^{n} \frac{\ell_k(x) \operatorname{sgn} \ell_k(y_n)}{w(x_k)}$$

of degree at most n. Evidently

$$|q_n(x)| \leqslant \lambda_n(x) \leqslant \|\lambda_n\| = q_n(y_n), \qquad x \in \mathbf{R},$$

that is, $\|q_n\| = \|\lambda_n\|$.

Suppose $|x_j| \geqslant a_n$. Then applying (2) to this q_n with $x = x_j$ and using $|q_n(x_j)| = 1$ we obtain

$$1 \leqslant e^{-c_1 n((x_j/a_n) - 1)^{3/2}} \|\lambda_n\|.$$

Hence, a simple rearrangement yields the statement of the proposition with $c = 1/c_1^{2/3}$.

In particular, if the Lebesgue constant is optimal, that is, $\lambda_n = O(\log n)$, then Proposition 1 gives

$$\max_{0 \leqslant k \leqslant n} |x_k| \leqslant a_n \left(1 + c_2 \left(\frac{\log \log n}{n}\right)^{2/3}\right).$$

For the construction of such system of nodes see [S].

In some situations, it is interesting to consider the case when the weighted fundamental polynomials

$$\frac{w(x)}{w(x_k)} \ell_k(x) \tag{3}$$

are uniformly bounded. This is the case when we consider Hermite–Fejér interpolation with bounded norm (see [S]), or we want to construct convergent Lagrange interpolation polynomials of degree at most $n(1+\varepsilon)$ (see [V]). Then, similarly to the above considerations, we obtain

PROPOSITION 2. *If the weighted fundamental function* (3) *belonging to the node of largest absolute value is uniformly bounded, then*

$$\max_{0 \leqslant k \leqslant n} |x_k| \leqslant a_n \left(1 + \frac{c_2}{n^{2/3}}\right).$$

REFERENCES

[LL] A. L. Levin and D. S. Lubinsky, Christoffel functions, orthogonal polynomials, and Nevai's conjecture for Freud weights, *Constr. Approx.* **8** (1992), 463–535.

[MS] H. N. Mhaskar and E. B. Saff, Where does the sup-norm of a weighted polynomial live?, *Constr. Approx.* **1** (1985), 71–91.

[S] J. Szabados, Weighted Lagrange and Hermite–Fejér interpolation on the real line, *J. Inequal. Appl.* **1** (1997), 99–123.

[V] P. Vértesi, A convergence theorem in weighted interpolation, I, *Acta Math. Hungar.*, in press.